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Translated by A. Y.

ON A CERTAIN CONVERGENCE GAME

PMM Vol. 34, №5, 1970, pp. 804-811
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(Received May 7, 1970)

A game problem on the convergence of controlled objects by the instant $t = \vartheta$ is considered in a fixed time interval $[t_0, \vartheta]$. It is assumed that the pursuing object is an inertial point and that the pursued object is inertialess. The problem of constructing the pursuer's optimal minimax strategy is considered. This strategy ensures the minimax of the distance between the objects at a given instant. It is proved that the mixed strategy of special form (derived in [3]) which operates within the framework of the mathematical apparatus of differential equations in contingencies is such a strategy.

1. Let us consider a differential game involving the two objects $m^{(1)}$ and $m^{(2)}$ moving in the horizontal plane q_1q_2. The motion of the pursuing object $m^{(1)}(y_1, y_2)$ controlled by the first player is described by the system of equations

$$y_1^{\cdot} = y_3, \qquad y_2^{\cdot} = y_4, \qquad y_3^{\cdot} = u_3, \qquad y_4^{\cdot} = u_4 \tag{1.1}$$

where the control vector $u = u^*(u_3, u_4)$ satisfies the inequality

$$(u_3^2[t] + u_4^2[t])^{1/2} \leqslant \mu \tag{1.2}$$

The pursued object $m^{(2)}$ (z_1, z_2) controlled by the second player moves according to the equations

$$z_1^{\cdot} = v_1, \qquad z_2^{\cdot} = v_2 \tag{1.3}$$

where the control vector $v = v\ (v_1,\ v_2)$ is subject to the restriction

$$(v_1^2\ [t] + v_2^2\ [t])^{1/2} \leqslant \nu \tag{1.4}$$

The game payoff γ is the distance between the objects $m^{(1)}$ and $m^{(2)}$ at the given instant $t = \vartheta$, i. e.

$$\gamma = [(y_1\ [\vartheta] - z_1\ [\vartheta])^2 + (y_2\ [\vartheta] - z_2\ [\vartheta])^2]^{1/2} \tag{1.5}$$

As we know, Eqs (1. 1), (1. 3) under restrictions (1 2), (1.4) describe the motion of objects in the game pursuit problems sometimes referred to as "isotropic missile" or "boy-and-crocodile" problems [1–3].

Let us introduce some notation . We denote the time in our ancillary discussion by τ; the time in which the original game takes place is t. The argument τ in the symbols for the functions describing the motions and controls in the ancillary problems appears in parentheses. The square brackets in which we place the argument t means that we are referring to the motions and controls actually realized in the course of the game. The set consisting of the four-dimensional vectors $u = u\ (u_1,\ u_2,\ u_3,\ u_4) = u^*\ (u_3,\ u_4)$ whose components u_1 and u_2 satisfy the condition $u_1 \equiv u_2 \equiv 0$ and whose components u_3 and u_4 are subject to restriction (1. 2) will be denoted by U_a. The set of vectors $v = v\ (v_1,\ v_2)$ whose components v_1 and v_2 are subject to restrictions (1. 4) will be denoted by V_+.

We assume that the controlling force u is formed by the feedback principle and that its value $u\ [t]$ realized at each present instant $t \in [t_0,\ \vartheta]$ is determined by the position realized at that instant.

We shall employ the definition of a game strategy given in [3]. Thus, the permissible pursuit strategies U will be identified with the sets $U_* = U_*\ (t,\ y,\ z)$ associated with each possible position $(t,\ y,\ z)$ and having the following properties:

1) the inclusions

$$U_*\ (t,\ y,\ z) \subset U_a$$

are fulfilled

2) the sets $U_*\ (t,\ y,\ z)$ are closed and convex;

3) the sets $U_*\ (t,\ y,\ z)$ are semicontinuous above by inclusion as $(t,\ y,\ z)$ varies in the neighborhood of each possible position.

In the case of the pursued object we permit any integrable realizations $v\ [t]$ restricted by condition (1. 4). The motion of system (1. 1), (1. 3) generated by the strategy $U = U\ (t,\ y,\ z)$ and the arbitrary integrable realization of the control $v = v\ [t]$ is defined as an absolutely continuous vector function $(y\ [t],\ z\ [t])$ which for almost all $t \in [t_0,\ \vartheta]$ satisfies the system of equations

$$y_1^{\cdot} = y_3\ [t],\ y_2^{\cdot} = y_4\ [t],\quad y_3^{\cdot}\ [t] = u_3\ [t],\ y_4^{\cdot}\ [t] = u_4\ [t]$$

$$z_1^{\cdot} = v_1\ [t],\quad z_2^{\cdot} = v_2\ [t]$$

where

$$u\ [t] \in U_*\ (t,\ y\ [t],\ z\ [t])$$

We can now formulate the problem as follows. We are to find the permissible strategy $U^\circ\ (t,\ y,\ z)$ which produces the minimum in the following relation:

$$\gamma^* = \min_U \max_{v[t]} \max_{y[t]} \gamma =$$

$$= \min_U \max_{v[t]} \max_{y[t]} [(y_1\ [\vartheta] - z_1\ [\vartheta])^2 + (y_2\ [\vartheta] - z_2\ [\vartheta])^2]^{1/2} \tag{1.6}$$

Here $\max_{y[t]}$ is taken over all the solutions $y\ [t]$ corresponding to the strategy $U = U\ (t,\ y,\ z)$ and to the control $v = v\ [t]$; $\min_U$ and $\max_v$ are computed over all the possible strategies $U = U\ (t,\ \ y,\ \ z)$ and integrable vector functions $v =$ $= v\ [t]\ (v\ [t] \in V_+)$, respectively.

2. Now let us turn to the solution of the stated problem. To this end we consider the attainability domain ([4], p. 331) of objects (1.1), (1.2) and (1.3), (1.4) from the position $(t,\ y\ [t])$ and $(t,\ z\ [t])$, respectively, by the instant $t = \vartheta$; we shall construct this domain in the plane q_1q_2. Let us denote by $G_\varepsilon^{(1)}\ (t,\ y,\ \vartheta)$ the closed ε-neighborhood of the domain $G^{(1)}\ (t,\ y,\ \vartheta)$. The domain $G^{(1)}\ (t,\ y,\ \vartheta)$ is the set of points lying in the plane q_1q_2 and attainable by the first player from the position $(t,\ y\ [t])$ by the instant $t = \vartheta$. Similarly, the attainability domain $G^{(2)}(t,\ z,\ \vartheta)$ is defined as the set of points lying in the plane q_1q_2 and attainable by the second player from the position $(t, z\ [t])$ by the instant $t = \vartheta$. The attainability domains $G_\varepsilon^{(1)}\ (t,\ y, \vartheta)$ and $G^{(2)}\ (t, z, \vartheta)$ in the plane q_1q_2 are disks. The radius $r_\varepsilon^{(1)}\ (t, \vartheta)$ of the domain $G_\varepsilon^{(1)}(t,\ \ y,\ \ \vartheta)$ is given by the equation

$$r^{(1)}\ (t,\ \ \vartheta) = {}^1/_2\mu\ (\vartheta - t)^2 + \varepsilon \qquad (\varepsilon \geqslant 0) \tag{2.1}$$

and the radius $r^{(2)}\ (t,\ \vartheta)$ of the domain $G^{(2)}\ (t,\ z,\ \vartheta)$ by the equation

$$r^{(2)}(t,\ \ \vartheta) = \nu\ (\vartheta - t) \tag{2.2}$$

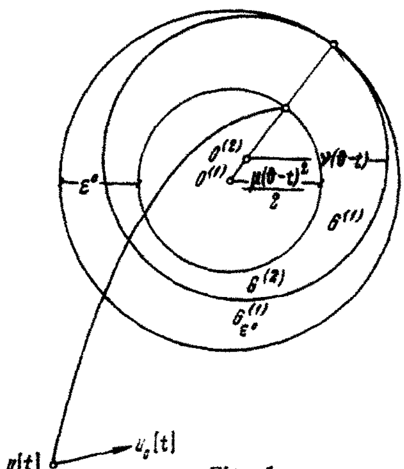

Fig. 1

The center $O^{(1)}$ of the domain $G_\varepsilon^{(1)}$ at the instant t has the coordinates

$$y_c = \{y_{1c}\ [t],\ y_{2c}\ [t]\} = \{y_1\ [t] + (\vartheta - t)y_3\ [t],\ y_2\ [t] + (\vartheta - t)y_4\ [t]\}$$

The coordinates of the center $O^{(2)}$ of the domain $G^{(2)}$ coincide with the coordinates of the point $\{z_1\ [t],\ z_2\ [t]\}$.

Let $\Delta = \Delta\ (t,\ \ y,\ \ z)$ be the distance between the centers $O^{(1)}$ and $O^{(2)}$ of the disks, so that

$$\Delta^2 = \{z_1\ [t] - y_1\ [t] - (\vartheta - t)\ y_3\ [t]\}^2 + \{z_2[t] - y_2[t] - (\vartheta - t)y_4\ [t]\}^2$$

Expressions clearly imply the possibility of choosing initial states $y\ [t_0]$ and $z\ [t_0]$ in such a way that it is impossible to retain the domain $G^{(2)}\ (t,\ z\ [t],\ \vartheta)$ in an arbitrarily small neighborhood of the domain $G_{\varepsilon^\circ}^{(1)}\ (t,\ y\ [t],\ \vartheta)$ for all $t \in [t_0,\ \vartheta]$, since the radius $r^{(1)}$ descreases as the square of the quantity $\vartheta - t$, the radius $r^{(2)}$ is on the order of the quantity $\vartheta - t$, and the initial value $\varepsilon^\circ = \varepsilon\ (t_0,\ \ y\ [t_0],\ \ z\ [t_0])$ can always be made arbitrarily small (even zero) by suitable choice of the initial position $(t_0,\ \ y_0 = y\ [t_0],\ \ z_0 = z\ [t_0])$.

Let us suppose that at some instant $t \in [t_0,\ \vartheta]$ the domain $G^{(2)}\ (t,\ z,\ \vartheta)$ is absorbed by the domain $G_{\varepsilon[t]}^{(1)}\ (t,\ y,\ \vartheta)$ (Fig. 1), where

$$\varepsilon\ [t] = \nu\ (\vartheta - t) - {}^1/_2\mu\ (\vartheta - t)^2 + \Delta\ (t,\ \ y,\ \ z) \tag{2.3}$$

Let us show that the required strategy $U^\circ\ (t,\ \ y,\ \ z)$ which yields $\min_U \max_v \max_y \gamma$

is the so-called "mixed" extremal strategy U^- defined by certain sets U_-. This strategy U^- is constructed as follows [3].

We know [3] that the function $\varepsilon^\circ(t,\ y[t],\ z[t])$ is defined by the equation

$$\varepsilon^\circ(t,\ y[t],\ z[t]) = \max_{\|l\|=1} [\rho^{(2)}(t,\ z,\ \vartheta,\ l) - \rho^{(1)}(t,\ y,\ \vartheta,\ l)] \tag{2.4}$$

where $\rho^{(2)}(t,\ z,\ \vartheta,\ l)$ and $\rho^{(1)}(t,\ y,\ \vartheta,\ l)$ are the support functions of the sets $G^{(2)}(t,\ z,\ \vartheta)$ and $G^{(1)}(t,\ y,\ \vartheta)$.

If the game position $(t,\ y[t],\ z[t])$ is such that the maximum in (2.4) is provided by the unique vector $l = l^\circ(t,\ y,\ z,\ \vartheta)$ of this position, we associate the set $U_* = U_-(t,\ y,\ z)$ with the set of all those vectors u_0 for which the maximum condition

$$l^{\circ\prime}(t, y, z, \vartheta) Y(\vartheta,\ t)\ u_0 = \max_{u\in U_a} l^{\circ\prime}(t,\ y,\ z,\ \vartheta) Y(\vartheta,\ t)\, u$$

holds.

Here $Y(\vartheta,\ t)$ is the fundamental matrix of system (1.1) for $u \equiv 0$, and the prime denotes transposition of the column vector l°.

In this case the set $U_-(t,\ y,\ z)$ consists of a unique vector,

$$U_-(t,\ y,\ z) = \{0,\ 0,\ \mu(z_1 - y_1 - (\vartheta - t)\, y_3)\,\Delta^{-1},\quad \mu(z_2 - y_2 - (\vartheta - t)\, y_4)\,\Delta^{-1}\}$$
$$(\Delta \neq 0)$$

If the maximum in (2.4) is not provided by a unique vector l° then we set

$$U_-(t,\ y,\ z) = U_a$$

We note that in the problem under consideration the vector l° is unique for $\Delta > 0$ and nonunique for $\Delta = 0$.

We have thus defined the strategy U^- given by the sets

$$U_-(t,\ y,\ z) = \{0,\ 0,\ \mu(z_1 - y_1 - (\vartheta - t)y_3)\,\Delta^{-1},\ \mu(z_2 - y_2 - (\vartheta - t)y_4)\,\Delta^{-1}\}$$
$$(\Delta > 0)$$
$$U_-(t,\ y,\ z) = U_a \qquad (\Delta = 0) \tag{2.5}$$

3. Let the duration of game (1.1)–(1.6) satisfy the condition $\vartheta - t_0 > \nu / \mu$.

Let us break up the interval $[t_0,\ \vartheta]$ into the interval $[t^*,\ \vartheta]$ and the half-interval $[t_0,\ t^*)$, i.e. at the point $t^* = \vartheta - \nu / \mu$.

We can verify the validity of the following ancillary statement.

For any initial position $(t^\circ, y[t^\circ], z[t^\circ])$, where $t^* \leqslant t^\circ \leqslant \vartheta$, we have

$$\min_U \max_{v[t]} \max_{y[t]} \gamma = r^{(2)}(t^\circ) - r^{(1)}(t^\circ) + \Delta(t^\circ,\ y[t^\circ],\ z[t^\circ])\ (\Delta \geqslant 0)$$

This minimax is provided by the strategy U^-.

Proof. Let the domain $G^{(2)}(t^\circ,\ z,\ \vartheta)$ be absorbed by the function $G^{(1)}_{\varepsilon[t^\circ]}(t^\circ,\ y,\ \vartheta)$, where the function $\varepsilon[t]$ is of the form (2.3), at the instant t°, $t^* \leqslant t^\circ < \vartheta$.

In the domain $(t,\ y,\ z)$ defined by the inequality $\Delta(t,\ y,\ z) > 0$ the derivative $d\Delta / dt$ computed along the system motion (1.1), (1.3) generated by the first-player strategy U^- and by the integrable second-player strategy $v[t]$, is of the form

$$d\Delta / dt = \{(z_1 - y_1 - (\vartheta - t)y_3)(v_1[t] - (\vartheta - t)u_3[t]) + \tag{3.1}$$
$$+ (z_2 - y_2 - (\vartheta - t)y_4)(v_2[t] - (\vartheta - t)\ u_4[t])\}\Delta^{-1} \leqslant \nu - \mu(\vartheta - t)$$

$$\nu - \mu(\vartheta - t) = \min_u \max_v d\Delta / dt \qquad (\Delta > 0) \tag{3.2}$$

where the control u which minimizes expression (3.2) coincides with the control dictated by the strategy U^- for $\Delta > 0$. The control $v^*[t]$ which maximizes the expression $d\Delta / dt$ with respect to v in the domain where $\Delta > 0$ is of the form

$$v^*[t] = \{\nu (z_1 - y_1 - (\vartheta - t)y_3)\Delta^{-1},\ \nu (z_2 - y_2 - (\vartheta - t)y_4)\Delta^{-1}\} \quad (\Delta > 0)$$

The function $\Delta(t, y, z)$ defined along system motion (1.1), (1.3) generated by the strategy U^- and by the arbitrary integrable second-player realization $v[t]$ is clearly a continuous function of the argument t.

Hence, the function $\Delta[t]$ is majorated as follows:

$$\Delta[t] \leqslant \int_{t^\circ}^{t} (\nu - \mu(\vartheta - \tau))\, d\tau + \Delta[t^\circ], \text{where } t^\circ \leqslant t \leqslant \vartheta \qquad (3.3)$$

From inequality (3.3) we infer that

$$\varepsilon[\vartheta] = r^{(2)}(\vartheta) - r^{(1)}(\vartheta) + \Delta[\vartheta] \leqslant \nu(\vartheta - t^\circ) - {}^1/_2\mu(\vartheta - t^\circ)^2 + \\ + \Delta[t^\circ] = r^{(2)}(t^\circ) - r^{(1)}(t^\circ) + \Delta[t^\circ] \qquad (3.4)$$

For $t = \vartheta$ we clearly have

$$\varepsilon(\vartheta, y[\vartheta], z[\vartheta]) = \{(y_1[\vartheta] - z_1[\vartheta])^2 + (y_2[\vartheta] - z_2[\vartheta])^2\}^{1/2} \qquad (3.5)$$

From (3.4) and (3.5) we have

$$\gamma \leqslant r^{(2)}(t^\circ) - r^{(1)}(t^\circ) + \Delta[t^\circ] \qquad (3.6)$$

Since, beginning at the instant $t = t^\circ$ and until the instant $t = \vartheta$, the second player can realize some constant control v of maximum absolute value which brings the point $z[t^\circ]$ to the intersection of the boundaries of the domains $G^{(1)}_{\varepsilon[t^\circ]}(t^\circ, y, \vartheta)$ and $G^{(2)}(t^\circ, z, \vartheta)$ and therefore ensures that

$$\gamma \geqslant r^{(2)}(t^\circ) - r^{(1)}(t^\circ) + \Delta[t^\circ]$$

at the instant $t = \vartheta$, it follows that the strategy $U = U^-$ is the optimal minimax first-player strategy in the interval $t^\circ \leqslant t \leqslant \vartheta$. This strategy ensures for the first player a game payoff not inferior to

$$\gamma = r^{(2)}(t^\circ) - r^{(1)}(t^\circ) + \Delta[t^\circ] \qquad (3.7)$$

Now let us consider the case where the equation $\Delta(t^\circ, y[t^\circ], z[t^\circ]) = 0$ is fulfilled at the instant $t = t^\circ$ In this case, making use of the continuity of the function $\Delta(t, y, z)$ in the arguments y and z and of the fact (implied by (3 2)) that

$$\min_u \max_v d\Delta / dt = \nu - \mu(\vartheta - t) > 0 \quad (t \in [t^\circ, \vartheta],\ \Delta > 0)$$

and repeating verbatim the above argument for the domain $\Delta > 0$ we conclude that the strategy U^- is the optimal minimax first-player strategy which ensures for the latter a game payoff not inferior to

$$\gamma = r^{(2)}(t^\circ) - r^{(1)}(t^\circ) \qquad (3.8)$$

From the above considerations we infer that if the duration of game (1.1)–(1.6) satisfies the condition $\vartheta - t_0 < \nu / \mu$, then the strategy U^- ensures for the first player a game payoff not inferior to (3.7) or (3.8) if $\Delta(t_0, y[t_0], z[t_0]) > 0$ or $\Delta(t_0, y[t_0], z[t_0]) = 0$, respectively.

4. Let us consider game (1.1)–(1.6) in the half-interval $[t_0, t^*)$. First we note that in accordance with (3.1) and (3.2) in the domain (t, y, z) where $\Delta(t, y, z) > 0$ the derivative $d\Delta / dt$ computed along system motion (1.1), (1.3) generated by the first-player strategy U^- and by the integrable second-player strategy $v[t]$ satisfies the condition $d\Delta / dt \leqslant \nu - \mu(\vartheta - t) < 0 \quad (t \in [t_0, t^*),\ \Delta > 0) \qquad (4.1)$

Let the first player choose a control dictated by the strategy U^- in the half-interval $[t_0, t^*)$; let the second player realize some integrable control $v[t]$. Then, depending on the values of the function $\Delta(t, y, z)$ computed along system motion (1.1), (1.3) generated by the first-player strategy U^- and by the arbitrary integrable realization of the second-player control v, we have three cases of game (1.1)–(1.6). Let us consider each of these cases individually and show that in each of these cases U^- is the first player's optimal minimax strategy.

The first case prevails when inequality $\Delta(t, y, z) > 0$ is fulfilled for all $t \in [t_0, t^*)$ along system motion (1.1), (1.3).

In this case, computing the derivative $d\varepsilon(t, y[t], z[t]) / dt$ along system motion (1.1), (1.3) for $t < t^*$, we find that $d\varepsilon(t, y[t], z[t]) / dt \leqslant 0$; next, repeating the statements used to prove the ancillary statement for $t \geqslant t^*$, we find that U^- is the optimal minimax strategy which ensures for the first player a game payoff γ not inferior to

$$\min_U \max_{v[t]} \max_{y[t]} \gamma = r^{(2)}(t^*) - r^{(1)}(t^*) + \Delta^* \leqslant \varepsilon^\circ(t_0)$$
$$\Delta^* = \Delta(t^*, y[t^*], z[t^*])$$

The second case occurs when $\Delta(t_0, y_0, z_0) = 0$ at the instant $t = t_0$. The ancillary statement implies that if the duration of game (1.1)–(1.6) satisfies the condition $\vartheta - t_0 > \nu / \mu$, then $\quad \min_U \max_{v[t]} \max_{y[t]} \gamma \geqslant {}^1/_2 \nu^2 \mu^{-1}$

But (4.1) implies that system motion (1.1), (1.3) generated by the first-player strategy U^- and by an arbitrary integrable realization of the second-player control v remains on the set $\Delta(t, y, z) \equiv 0$ for $t \in [t_0, t^*)$. Hence, our preceding discussion implies that the strategy U^- ensures for the first player a payoff of game (1.1)–(1.6) which is not inferior to

$$\min_U \max_{v[t]} \max_{y[t]} \gamma = {}^1/_2 \nu^2 \mu^{-1}$$

and is the optimal minimax first-player strategy in this case.

Finally, we have the third case of game (1.1)–(1.6) where the inequality $\Delta(t_0, y_0, z_0) > 0$ is fulfilled for some $t = t_0$, and where the equation $\Delta(t_0^*, y_0^*, z_0^*) = 0$ (where $y_0^* = y[t_0^*]$, $z_0^* = z[t_0^*]$) is satisfied for the first time at some $t = t_0^*$ $(t_0 < t_0^* \leqslant t^*)$ along the system motion (1.1), (1.3) under consideration.

Let us show that U^- is the optimal minimax first-player strategy in this case as well.

Recalling that the duration of game (1.1)–(1.6) satisfies the condition $\vartheta - t_0 > \nu / \mu$, we infer from the ancillary statement that

$$\min_U \max_{v[t]} \max_{y[t]} \gamma \geqslant {}^1/_2 \nu^2 \mu^{-1} \qquad (4.2)$$

Expressions (3.2) and (4.1) imply that the strategy U^- ensures fulfillment of the equation

$$\Delta_0^* = \min_U \max_{v[t]} \max_{y[t]} \Delta(t^*_0, y[t_0^*], z[t_0^*]) = 0$$

by the instant $t = t_0^*$.

But (4.1) implies that system motion (1.1), (1.3) generated by the first-player strategy U^- and by the arbitrary integrable second-player realization $v[t]$ remains (during the interval $t_0^* \leqslant t \leqslant t^*$) on the set (t, y, z) defined by the condition $\Delta(t, y, z) \equiv 0$ for $t \in [t_0^*, t^*]$.

Thus, the strategy U^- ensures equality in (4.2) for the first player. This means that the strategy U^- ensures for the first player a payoff not inferior to

$$\min_U \max_{v[t]} \max_{y[t]} \gamma = {}^1/_2 \nu^2 \mu^{-1}$$

and constitutes the optimal minimax strategy in this case of the game.

The resulting solution $(y[t], z[t])$ of system (1.1), (1.3) corresponding to the strategy $U = U^- = U^\circ$ and to an arbitrary integrable realization $v = v[t]$ can be approximated by means of the following discrete scheme.

Let us break up the interval $[t_0, \vartheta]$ into half-intervals $\delta = [\tau_i, \tau_{i+1})$ at the points τ_i $(0 \leqslant i \leqslant n)$. The discrete strategy $U^{\circ(\delta)}(\tau_i, y[\tau_i], z[\tau_i])$ forms the control $u[t]$ in the following way. Let the position $(\tau_i, y[\tau_i], z[\tau_i])$ be realized at the instant $t = \tau_i$. If $\Delta(\tau_i, y[\tau_i], z[\tau_i]) > 0$, then we set the control $u[t]$ $(\tau_i \leqslant t < \tau_{i+1})$ constant and equal to

$$u_i[t] = \{0, 0, \mu(z_1[\tau_i] - y_1[\tau_i] - (\vartheta - \tau_i) y_3[\tau_i]) \Delta^{-1}[\tau_i]$$

$$\mu(z_2[\tau_i] - y_2[\tau_i] - (\vartheta - \tau_i) y_4[\tau_i]) \Delta^{-1}[\tau_i]\}$$

in the half-interval $[\tau_i, \tau_{i+1})$.

But if $\Delta(\tau_i, y[\tau_i], z[\tau_i]) = 0$, then we set the control $u[t]$ $(\tau_i \leqslant t < \tau_{i+1})$ equal to an arbitrary constant vector which satisfies the condition $u[t] = u[\tau_i] \in U_a$. The second player acts as before, choosing arbitrary integrable realizations $v[t]$.

We can show that as δ tends to zero, the resulting trajectories $(y[t], z[t])$ of system (1.1), (1.3) converge to a certain solution of the equations in contingencies corresponding to the strategy U° and to the arbitrary integrable realization $v[t]$.

We are therefore able to say that for any $\alpha > 0$ there exists a $\delta^\circ > 0$ such that for any discrete scheme with the interval δ ($0 < \delta \leqslant \delta^\circ$) and for any second-player behavior we have a game payoff $\gamma^{(\delta)}$ which satisfies the inequality

$$\gamma^{(\delta)} \leqslant \min_U \max_{v[t]} \max_{y[t]} \gamma + \alpha$$

The author is grateful to N. N. Krasovskii for stating the problem and for his useful criticism of the results, and to A. I. Subbotin for his interest and valuable comments.

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Translated by A. Y.